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Physical Structure of the Energy-Momentum Tensor of a Dust of Spinning Particles

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The algebraic classification of the energy-momentum tensor of a dust of spinning particles in Einstein–Cartan theory based on the Segré type is presented.

1. INTRODUCTION

Recently Hall and Negm (1986) presented a detailed study of the physical structure of the energy-momentum tensor (EMT) in general relativity. They examined the dominant energy condition (Hawking and Ellis, 1973) of various energy-momentum tensors and its implications for the eigenvalue problem. One of the consequences of the application of the dominant energy condition to the EMT is that they forbid the EMT from having Segré type $\{3\ 1\}$ or $\{z\ \bar{z}\ 11\}$, where the numerals inside the brackets refer to the multiplicity of the real eigenvalue represented and the $z\bar{z}$ pair in the last symbol refers to a pair of complex conjugate eigenvalues of multiplicity one in the Segré notation (Hall, 1976; Kramer *et al.*, 1980). In this paper we prove that the last restriction does not apply when one considers an energy-momentum tensor of a dust of spinning particles in Einstein–Cartan theory (Hehl *et al.*, 1976, Trautman, 1980). Since it is enough to find an example that possesses complex conjugate eigenvalues, we consider here for reasons of simplicity the case of the EMT of a dust of spinning particles in a cosmological model with torsion free of expansion (Raychaudhuri, 1979).

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2. THE EIGENVALUE PROBLEM IN EINSTEIN–CARTAN THEORY

The algebraic classification of the EMT in Einstein theory was initiated by Petrov (1969) and was based on the multiplicity of the eigenvalues which were obtained by solving the eigenvalue equation

$$\det(T_{ij}^{\mathrm{GR}} - \lambda_{ij}) = 0 \tag{1}$$

where λ represents the eigenvalues and det indicates the determinant of the matrix inside the brackets. We propose to consider in equation (1) instead of the general relativistic (GR) EMT T_{ij}^{GR}, the nonsymmetric energy-momentum tensor of EC theory given by

$$T_{ij}^{\mathrm{EC}} = T_{ij}^{\mathrm{GR}} + \tfrac{1}{2}\nabla_k^r S_{ij}^k \tag{2}$$

where S_{ij}^k is the spin angular momentum density related to the torsion of the spacetime and ∇_k^r represents the Riemann–Cartan connection of a spacetime with spin density. Since S_{ij}^k obeys the Frenkel conditions

$$\begin{aligned} S_{ij}u^k &= S_{ij}^k \\ S_{ik}u^k &= 0 \end{aligned} \tag{3}$$

where S_{ij} is the spin angular momentum tensor and u^k is the 4-velocity of the spinning fluid, we can apply equation (3) to simplify the EMT (2). After some algebra we obtain

$$T_{ij}^{\mathrm{EC}} = T_{ij}^{\mathrm{GR}} + \tfrac{1}{2}(\theta S_{ij} + \dot{S}_{ij}) \tag{4}$$

where the dot indicates the operator $(\cdot) \equiv u^k \nabla_k^r$ and $\theta \equiv \nabla_k^r u^k$ is the EC expansion parameter.

From equation (4) we can obtain the final eigenvalue equation

$$\det[T_{ij}^{\mathrm{GR}} + \tfrac{1}{2}(\theta S_{ij} + \dot{S}_{ij}) - \lambda g_{ij}] = 0 \tag{5}$$

To simplify the final characteristic polynomial and find the roots (the eigenvalues) we adopt the following assumptions:

(i) The EMT T_{ij}^{GR} is considered to be the EMT of a spinning dust or

$$T_{ij}^{\mathrm{GR}} = \rho u_i u_j \tag{6}$$

where ρ is the energy density of the spinning matter.

(ii) The spinning dust is expansion free, or $\theta = 0$.

(iii) Because λ is a scalar, we can simplify our calculations if (Kolassis *et al.*, 1988) in the event of the spacetime under consideration, we use a locally Minkowskian coordinate system where

$$g_{ij} = \rho_{ij} \tag{7}$$

where η_{ij} is the Minkowski metric given by $\eta_{ij} = \mathrm{diag}(+1, -1, -1, -1)$.

(iv) We also suppose that the spins of the particles are aligned along the z axis of symmetry, which means that the only nonvanishing component of S_{ij} is S_{12}.

Using these four hypotheses in equation (5), we can write in matrix form

$$\det\begin{bmatrix} \rho-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & \tilde{s} & 0 \\ 0 & -\tilde{s} & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} = 0 \tag{8}$$

where $\tilde{s} \equiv \dot{S}_{12}/2$ is the only nonvanishing component of the spin.

Expanding the determinant in equation (8), we get

$$(\rho-\lambda) \cdot [\lambda^2 + \tilde{s}^2] \cdot \lambda = 0 \tag{9}$$

Equation (9) can be decomposed in the canonical polynomial form

$$\lambda \cdot (\lambda - \rho) \cdot (\lambda - i\tilde{s})(\lambda + i\tilde{s}) = 0 \tag{10}$$

which allows us to identify the roots of the polynomial (eigenvalues) very easily as

$$\lambda_1 = \rho; \qquad \lambda_2 = i\tilde{s}, \qquad \lambda_3 = -i\tilde{s}, \qquad \lambda_4 = 0 \tag{11}$$

which proves that we have a pair of complex conjugate roots $\lambda_1 = \bar{\lambda}_2$ which corresponds to the Segré type $\{z\bar{z}11\}$. In the torsion-free case expression (11) reduces to the well-known GR case (Hawking and Ellis, 1973).

3. DISCUSSION

A more detailed study of the algebraic classification of the EMT in EC theory can be accomplished by the decomposition of this tensor along a null tetrad basis $(m, \bar{m}, l, n)$ and by the study of the eigenvectors. This appears elsewhere (Garcia de Andrade and Smalley, 1991). An analogous study of the algebraic structure of the Riemann–Cartan spacetime can be done for the Ricci tensor in U_4 (Garcia de Andrade, 1989). Nevertheless a Petrov classification of the Weyl tensor in U_4 presents some difficulties due to some changes in the symmetries of the Weyl tensor in U_4 (Garcia de Andrade, 1988).

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